Semigroup of positive maps for qu-dit states and entanglement in tomographic probability representation

V. I. MAN'KO, G. MARMO, A. SIMONI,
F. VENTRIGLIA

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Tomographic Papers

V. Man'ko, G. Marmo, A. Simoni, A. Stern,
G. Sudarshan, F. Ventriglia

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Introduction

Probabilistic description of a physical system: states $\mathcal{S}$ and observables $\mathcal{O}$

a pairing $\mu : (\rho, A) \in \mathcal{S} \times \mathcal{O} \to \mu_{A,\rho}$

$\in$ {Borel probability measures on $\mathbb{R}$}

Borel set $E \subseteq \mathbb{R} \rightsquigarrow \mu_{A,\rho}(E) \in [0,1]$ probability that value of A in the state ρ is in E.

Subset $\tau \in \mathcal{O}$ tomographic $\Leftrightarrow$ measures $\left\{\mu_{A,\rho}\right\}_{A\in\tau}$ identify state ρ. Group $\mathcal{G}$ acting on fiducial A_0 generates τ.

States for (2j+1)-levels Q. systems : spin-j states (qudits) $\rightsquigarrow (2j+1) \times (2j+1)$-density matrices ρ.

Spin Tomography: choose $\tau = \left\{U^\dagger J_z U\right\}$

$\mathcal{G}$=rotation group, U unitary irr. rep. matrix, fiducial op. $J_z = \sum_{m=-j}^{j} m \left|m\right\rangle \left\langle m\right|$ generator.

Tomogram: $\mathcal{W}_\rho(m, U)$ = value of concentrated measure $\mu_{U^\dagger J_z U,\rho}$ at the spectral point m :

$\mathcal{W}(m, U) = \mathrm{Tr}(U^\dagger \left|m\right\rangle \left\langle m\right| U\rho) = \left\langle m \middle| U\rho U^\dagger \middle| m \right\rangle .$

In the talk will be discussed:

1) The geometry of qudit states as that of a simplex and the set of positive maps of qudit states as stochastic and bistochastic matrices moving points on the simplex.

2)The relation of stochastic and bistochastic semigroups with Lie groups. A class of positive maps acting transitively on quantum states is introduced by relating stochastic and quantum stochastic maps in the tomographic setting.

3) The relation of stochastic matrices with Bell inequality violation for entangled states of two qubits.

Tomogram of a qutrit density matrix

The density matrix of a qutrit (spin-1) state (trace 1, hermitian 3×3 matrix) reads:

$$\rho = U_0 \tilde{\rho} U_0^\dagger \ , \ \ \tilde{\rho} = \begin{pmatrix} \tilde{\rho}_1 & 0 & 0 \\ 0 & \tilde{\rho}_2 & 0 \\ 0 & 0 & \tilde{\rho}_3 \end{pmatrix}$$

Eigenvalues: $\tilde{\rho}_k \geq 0$.

Eigenvec.s: columns of unitary $U_0 = ||\vec{u}_1, \vec{u}_2, \vec{u}_3||$

Fix phases and ordering so that

$\rho \vec{u}_1 = \tilde{\rho}_1 \vec{u}_1 \ , \ \rho \vec{u}_2 = \tilde{\rho}_2 \vec{u}_2 \ , \ \rho \vec{u}_3 = \tilde{\rho}_3 \vec{u}_3 \ .$

Probability vector

$$\vec{\tilde{\rho}} = \begin{bmatrix} \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{bmatrix}, \ \sum_{k=1}^{3} \tilde{\rho}_k = 1 \ , \ 0 \leq \tilde{\rho}_k \leq 1 \ ,$$

a point on the triangle with vertices in $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, (2-dim simplex in $\mathbb{R}^3$).

Linear maps on $\vec{\tilde{\rho}}$:

$$\vec{\tilde{\rho}} \longrightarrow \vec{\tilde{\rho}}' = M \vec{\tilde{\rho}}$$

are stochastic matrices M :

$$e_0^\top M = e_0^\top \ , \ e_0^\top = (1,1,1) \ , \ M_{kh} \geq 0 \ .$$

Stochastic matrices form a semigroup. Any convex sum of them is a stochastic matrix. So triangle (qutrit simplex) is invariant under the action of a linear stochastic map. Point $M\vec{\tilde{\rho}}$ belongs to the same triangle.

Qutrit Tomograms $\mathcal{W}_\rho(m,U)$ are probability vector components

$$\begin{aligned}
\mathcal{W}_1(U) &= \mathcal{W}(+1,U) = \left\langle 1 \left| U U_0 \tilde{\rho} U_0^\dagger U^\dagger \right| 1 \right\rangle \ , \\
\mathcal{W}_2(U) &= \mathcal{W}(0,U) = \left\langle 0 \left| U U_0 \tilde{\rho} U_0^\dagger U^\dagger \right| 0 \right\rangle \ , \\
\mathcal{W}_3(U) &= \mathcal{W}(-1,U) = \left\langle -1 \left| U U_0 \tilde{\rho} U_0^\dagger U^\dagger \right| -1 \right\rangle \ .
\end{aligned}$$

Direct calculation shows that :

$$\mathcal{W}_k(U) = \sum_{h=1}^{3} |(UU_0)_{kh}|^2 \, \tilde{\rho}_h \Leftrightarrow \vec{\mathcal{W}}(U) = M\vec{\tilde{\rho}}$$

where $M_{kh} = |(UU_0)_{kh}|^2$.
UU_0 unitary $\Longrightarrow M$ bistochastic matrix:

$$e_0^\top M = e_0^\top \ , \ M\vec{e}_0 = \vec{e}_0 \ .$$

The qutrit state is determined by a bistochastic map acting on the probability vector:
$\vec{\mathcal{W}}(U) = M\vec{\vec{\rho}}$
Geometrically, qutrit states are the orbit of the unitary group acting on triangle (qutrit simplex). The unitary group action is *via* the action of bistochastic maps.

Relation to Lie groups: restrict to invertible stochastic and bistochastic maps. Leave out nonnegativity of their entries. Then they form groups:

$$e_0^\top = e_0^\top M M^{-1} = e_0^\top M^{-1}$$
$$\vec{e}_0 = M^{-1} M \vec{e}_0 = M^{-1}\vec{e}_0$$

Qutrit case: bistochastic group $\sim$ $GL(2,\mathbb{R})$.
Choose a rotation $\mathcal{O}$ such that

$$\frac{1}{\sqrt{3}}\mathcal{O}\vec{e}_0 = \frac{1}{\sqrt{3}}\mathcal{O}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Rotate bistochastic matrices:

$$\tilde{M} = \mathcal{O} M \mathcal{O}^{\mathsf{T}} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 4−parameter group of $\tilde{M}$ is $GL(2,\mathbb{R})$.
$\det \mathcal{O} \neq 0 \Rightarrow$ set of matrices $\tilde{M} \sim$ bistochastic group matrices M.

For the stochastic matrices:

$$\tilde{M} = \mathcal{O} M \mathcal{O}^{\mathsf{T}} = \begin{pmatrix} A & B & m \\ C & D & n \\ 0 & 0 & 1 \end{pmatrix}$$

The group of these matrices is $IGL(2,\mathbb{R})$.

Positive maps

The problem of constructing positive maps, the dynamical maps providing a density matrix which could appear in a process of quantum evolution has a long history.

For finite $n-$level Q. systems:

Probability measures are probability vectors:

$\{\vec{v}: \ v_m \geq 0 \ \forall m = 1, \ldots, n; \ \sum_m v_m = 1\}$.

States are $n \times n-$density matrices ρ.

Dynamical maps: $\rho_0 \rightarrow \ \rho(t) = \mathbb{M}(t)\rho_0$, with $\mathbb{M}$ a $n^2 \times n^2-$matrix.

Dynamical maps on states ρ (Q. stochastic) induce dynamical maps on probability vectors $\vec{v}$ (stochastic matrices). We relate these two kinds of maps, in the tomographic setting. This is not standard approach where Q. stochastic maps are projection of isometries.

Parametrize a density state by a pair

$$\rho \rightarrow (U_0, \vec{\tilde{\rho}}) \ , \ U_0 = ||\vec{u}_1, \vec{u}_2, \vec{u}_3||$$

$$\rho\vec{u}_1 = \tilde{\rho}_1\vec{u}_1 \ , \ \rho\vec{u}_2 = \tilde{\rho}_2\vec{u}_2 \ , \ \rho\vec{u}_3 = \tilde{\rho}_3\vec{u}_3 \ .$$

This parametrizes Q. stochastic maps $\rho \to \rho'$ *via* unitary maps and stochastic maps. It is a different parametrization w.r.to those normally used in the literature.

In the tomographic framework density states are mapped onto probability vectors:
$\rho \to \vec{\mathcal{W}}(U) = |UU_0|^2 \,\vec{\vec{\rho}}$
We introduce positive maps of density states, parametrized by V unitary and M stochastic matrices

$$|UU_0|^2 \,\vec{\vec{\rho}} \;\to\; \left|UU_0'\right|^2 \vec{\vec{\rho}}' = |UVU_0|^2 \, M\vec{\vec{\rho}} \;,$$

$$(U_0, \vec{\vec{\rho}}) \to (U_0', \vec{\vec{\rho}}') = (VU_0, M\vec{\vec{\rho}}) \;.$$

Any density matrix ρ is determined by $(U_0, \vec{\vec{\rho}})$. Left actions on the unitary group and on the simplex by stochastic maps are transitive. So above positive maps act transitively on density matrices.
Our maps result convex linear:

$$\rho_1 \to \rho_1' \;;\; \rho_2 \to \rho_2' \Rightarrow \lambda_1\rho_1 + \lambda_2\rho_2 \to \lambda_1\rho_1' + \lambda_2\rho_2'$$
$$\lambda_1 + \lambda_2 = 1 \;,\; 0 \le \lambda_1, \lambda_2 \le 1$$

Entanglement

Consider two qubit states with state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|+\frac{1}{2}\right\rangle \left|-\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\right\rangle \left|+\frac{1}{2}\right\rangle\right).$$

The density matrix reads

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ,$$

Its eigenvalues and eigenvectors yield

$$\overrightarrow{\tilde{\rho}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \ , \ U_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

so the tomographic probability vector is:

$$\vec{\mathcal{W}}(U) = \frac{1}{2}\begin{bmatrix} |u_{12} + u_{13}|^2 \\ |u_{22} + u_{23}|^2 \\ |u_{32} + u_{33}|^2 \\ |u_{42} + u_{43}|^2 \end{bmatrix} \ .$$

The u's are the matrix elements of U.

To recognize the entanglement of the state with this tomogram, calculate the stochastic matrix M with columns four probability vectors $M = ||\vec{\mathcal{W}}(U_{ab}), \vec{\mathcal{W}}(U_{ac}), \vec{\mathcal{W}}(U_{db}), \vec{\mathcal{W}}(U_{dc})||$.
Each unitary matrix U_{hk} is a tensor product of two 2×2 unitary matrices $U_h \otimes U_k$:
$U_{hk} = U_h \otimes U_k \ , \ (h = a, d \ ; \ k = b, c) \ .$

Eventually, the stochastic matrix M is

$$M = \begin{pmatrix} x_{ab} & x_{ac} & x_{db} & x_{dc} \\ \frac{1}{2} - x_{ab} & \frac{1}{2} - x_{ac} & \frac{1}{2} - x_{db} & \frac{1}{2} - x_{dc} \\ \frac{1}{2} - x_{ab} & \frac{1}{2} - x_{ac} & \frac{1}{2} - x_{db} & \frac{1}{2} - x_{dc} \\ x_{ab} & x_{ac} & x_{db} & x_{dc} \end{pmatrix} ,$$

where x_{hk} is a function of the Euler angles determining U_h, U_k .

The Bell number B satisfying Bell's inequality

$$B \leq 2$$

can be calculated by means of a trace:

$$B = \mathrm{Tr}(ME) \ , \ \ E = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \ .$$

Elements of M are functions of four directions. Cirelson bound for Bell number is $2\sqrt{2}$, can be achieved in entangled states only. The bound is achieved when

$x_{ab} = x_{ac} = x_{db} = x \ ; \ x_{dc} = 1 - x \ .$

This gives

$$2\sqrt{2} = 4(4x-1) \ \Rightarrow \ x = \frac{2+\sqrt{2}}{8} \ .$$

Then universal stochastic matrix of an arbitrary, maximally entangled two-qubit state is

$$M = \begin{pmatrix} \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} \\ \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} \\ \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} \\ \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} \end{pmatrix} \ .$$

Conclusions

We resume main points.

We established a connection of spin tomograms with stochastic maps acting on a simplex . Stochastic and bistochastic $n \times n$ matrices have dense intersections with Lie groups $IGL(n-1, \mathbb{R})$ and $GL(n-1, \mathbb{R})$ respectively.

We constructed positive maps of density states parametrized by pairs of unitary matrices and stochastic matrices.

In the tomographic framework, for entangled two qubit states the Cirelson bound for the Bell number is associated with a universal stochastic matrix.

Finally, we hope to clarify the relations between the constructed positive map and other existing approaches in a future work.