Topology & quantum computing

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Outline

- Topology & fault-tolerant universal quantum computing
- Exchange properties & generalized statistics (**anyons**)

(*Chern-Simons framework*)

- Basic unitary transformations (**topological gates**)

B- and F-matrix (**B**raiding & **F**usion)

- Role of the constraints among **B**'s and **F**'s
  (topological invariance, solvability or integrability of the underlying quantum systems, computability)

Motivation & theoretical context

- **Topological quantum computation** (Abelian & non-Abelian anyons): Kitaev 1997; Preskill; Mochon;… (Chern-Simons Quantum Field Theory):Freedman et al 2000_; (both): Das Sarma et al. 0707.1889
- **Holonomic q-computation**: Zanardi & Rasetti 1999 _

rely on the idea that fault-tolerance can be improved in those quantum systems where information storage and manipulation are protected from local decoherence with no need of error-correction _ 'topological' ground states and unitary gates satisfying constraints that ensure invariance under natural topological transformations

In the same spirit, but looking at the basic combinatorial & algebraic structure of quantum Chern-Simons theory

- Applications of the spin network setting of topological q-computation
  - Efficient quantum algorithms for problems in geometric topology
  - Combinatorial approach to topological phases in 2D lattice models

- A. Marzuoli & M. Rasetti Ann. Phys. **318** (2005) 345
- S.Garnerone, A. Marzuoli & M. Rasetti (2006-2007), REFs in quant-ph/0703037
- Z. Kadar, A.Marzuoli & M.Rasetti 0806.3883 [math-ph]

Exchange properties of quantum particles (fields)

In 3 space dimensions identical particles are either boson or fermions
under mutual exchange

|_ (1,2) > _ |_ (2,1) > = +|_ (1,2) >

_ |_ (2,1) > = - |_ (1,2) >

In 2 space dimensions non trivial phase factors under mutual exchange may occur

|_ (1,2) > _ |_ (2,1) > = exp { i _} |_ (1,2) >

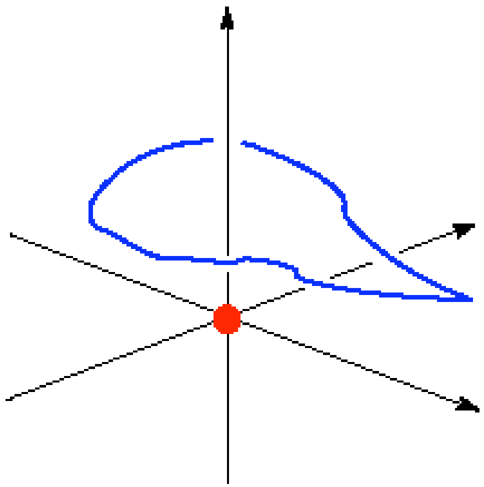
(cfr. the Aharonov-Bohm effect in elecrodynamics, but the analogy might be misleading)

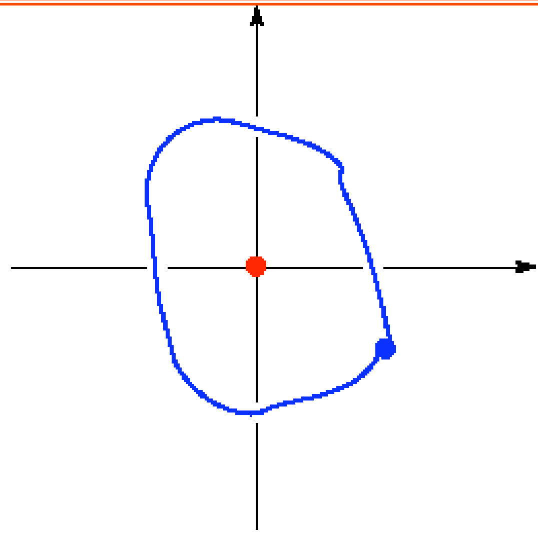
Equivalently, in terms of paths traveled by 'particle' _ around 'particle' _
(_ is a singularity in the domain)

D=3: **each path** can be continuously deformed to the trivial path={point}

D=2: the path cannot be continuously deformed to the trivial path={point}

All paths in this '**homotopy class**' **(*)** are characterized only by a positive [negative] integer = **winding number** of anticlockwise [clockwise] contours.
(*) neither the length nor the particular form of the curves do matter





Proper setting: anyonic field theories

Fock realization of N−component fields whose annihilation parts obey the relation

$$a_\alpha(x_1)a_\beta(x_2) = \mathcal{R}^{\delta\gamma}_{\beta\alpha}(\mathrm{x}_1,\mathrm{x}_2)\, a_\gamma(x_2)a_\delta(x_1)$$

$x_1, x_2 \in \mathbb{R}^s$ (spectral parameters);

$\mathcal{R}$: **exchange 'factor'**

($N^2 \times N^2$ matrix function)

- x-independent $\mathcal{R}$-matrix:
permutation group statistics
- piecewise-constant $\mathcal{R}$-matrix: braid statistics
- Unifying approach and more general x-dependent cases: A Liguori, M Mintchev (1995)

Spectral quantum Yang–Baxter equation

$$\mathcal{R}_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(\mathrm{x}_1,\mathrm{x}_2)\,\mathcal{R}_{\gamma_1\gamma_2}^{\beta_1\beta_2}(\mathrm{x}_2,\mathrm{x}_1) \;=\; \delta_{\alpha_1}^{\beta_1}\,\delta_{\alpha_2}^{\beta_2}$$

$$\bar{\mathcal{R}}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\mathrm{x}_1,\mathrm{x}_2) \;=\; \mathcal{R}_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\mathrm{x}_2,\mathrm{x}_1)$$

$$\mathcal{R}_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(x_1,x_2)\,\mathcal{R}_{\gamma_2\alpha_3}^{\gamma_3\beta_3}(x_1,x_3)\,\mathcal{R}_{\gamma_1\gamma_3}^{\beta_1\beta_2}(x_2,x_3) \;=$$

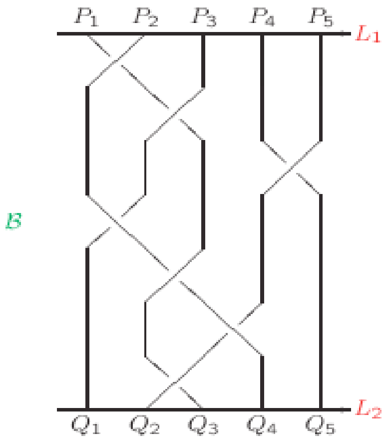
$$\mathcal{R}_{\alpha_2\alpha_3}^{\gamma_2\gamma_3}(x_2,x_3)\,\mathcal{R}_{\alpha_1\gamma_2}^{\beta_1\gamma_1}(x_1,x_3)\,\mathcal{R}_{\gamma_1\gamma_3}^{\beta_1\beta_3}(x_1,x_2)$$

- x_1, x_2, x_3 in a spectral set $\mathbb{R}^s$
- quantum fields with N components $\rightarrow\rightarrow\rightarrow$

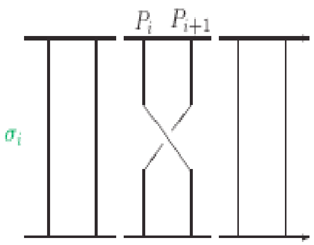
each exchange factor $\mathcal{R}$ is an $N^2 \times N^2$ **unitary** matrix–valued function on $\mathbb{R}^s \times \mathbb{R}^s$

Braids: generators & relations

The braid group $\mathcal{B}_n$

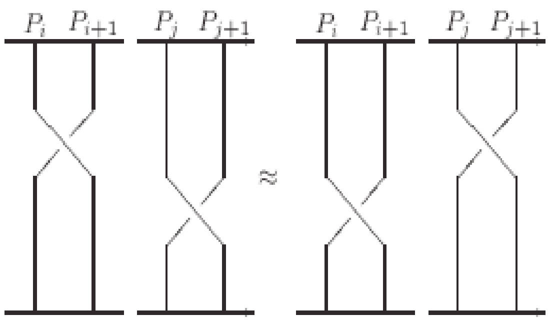


Elements of the braid group: weaving patterns



$i = 1, \ldots, (n-1)$

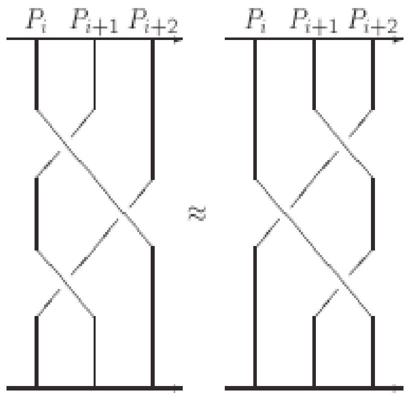
The braid group: generators



The braid group: relations

$\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$

Yang-Baxter relation



The braid group: relations

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall i = 1, \ldots, n-1$

Basic ingredients of anyonic computation

1. A finite set of particle types, i.e. labels specifying the possible values of the "charges". These are conserved charges that cannot be changed by local interactions

2. Particles can '**fuse**' and '**split**' according to a set of rules that give the charge of a composed particle in terms of the constituents

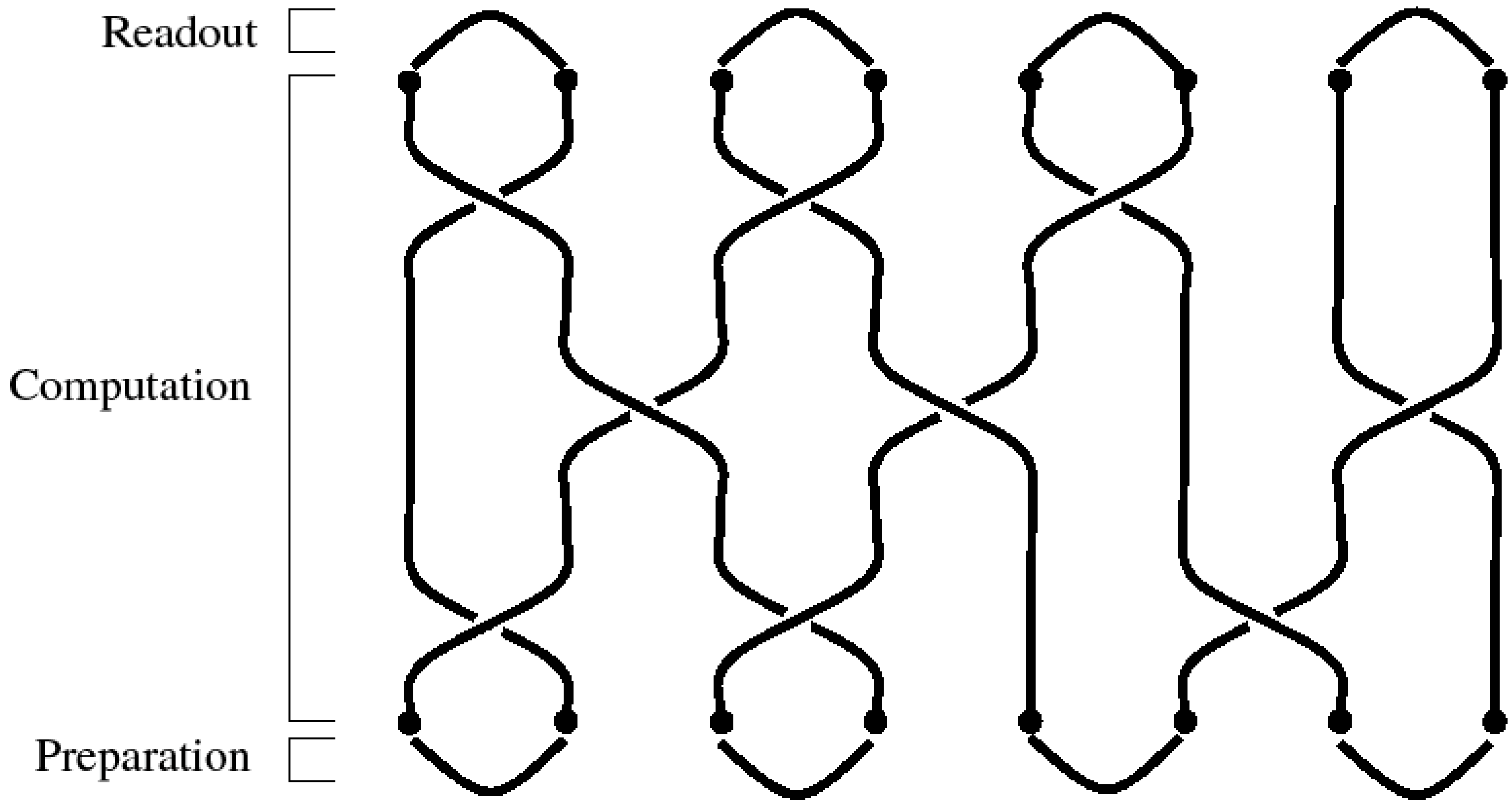
3. Particle trajectories are '**braided**' according to rules specifying how pairs (or bipartite subsystems) behave under exchange

Topological transformations that leave invariant braided trajectories will be recognized as algebraic constraints (compatibility conditions):

* Yang-Baxter or **hexagon equation** (relating 3 braidings and 3 fusions)
* **pentagon equation** (relating 5 fusions)

Computing with anyons

(creation/annihilation of particle-antiparticle pairs needed in preparation and readout phases)



Universal quantum computation

* **C**onformal **F**ield **T**heory
(conformal invariant models in 1+1 space-time dimensions)
* Charged sectors of 2+1 dimensional gauge theories
(with a finite or a compact Lie gauge group)

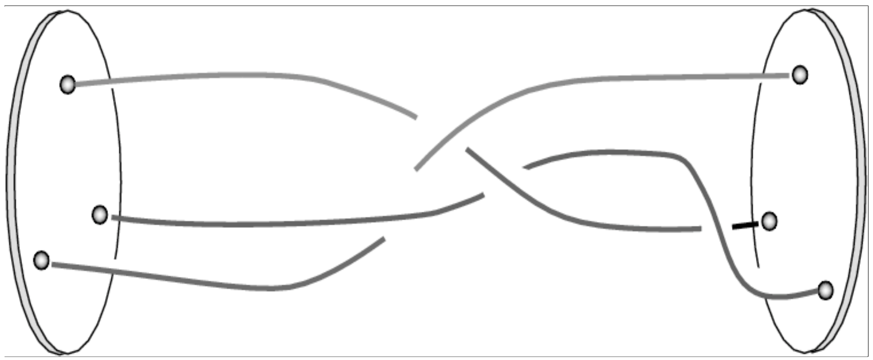
support Abelian and non-Abelian unitary representations of the **braid group** obeying generalized statistics **('anyons')**

Universality: achieved when every unitary gate can be approximated (within any precision) by **B**raiding & **F**using transformations, *i.e.* B and F are dense in the proper unitary groups

Universal models of Topological Computation:
-Fibonacci model (Kitaev)
-Chern-Simons theory for $SU(2)_k$ with $k \geq 3$

Quasi-particles trajectories in a (2+1)-dimensional Chern-Simons environment

Monodromy representations of the braid group in the associated boundary Conformal Field Theory



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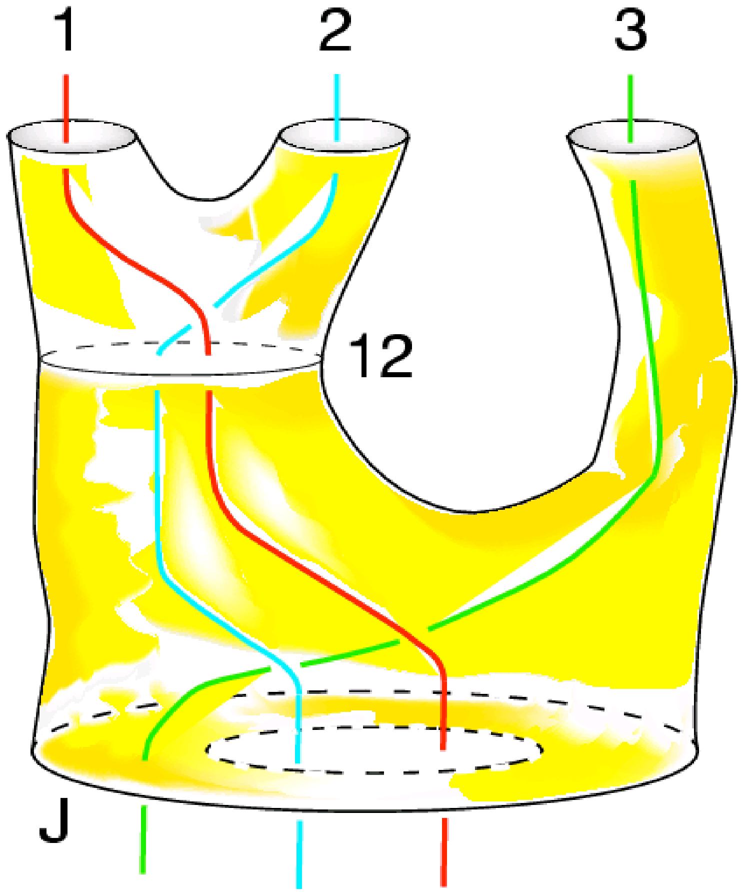

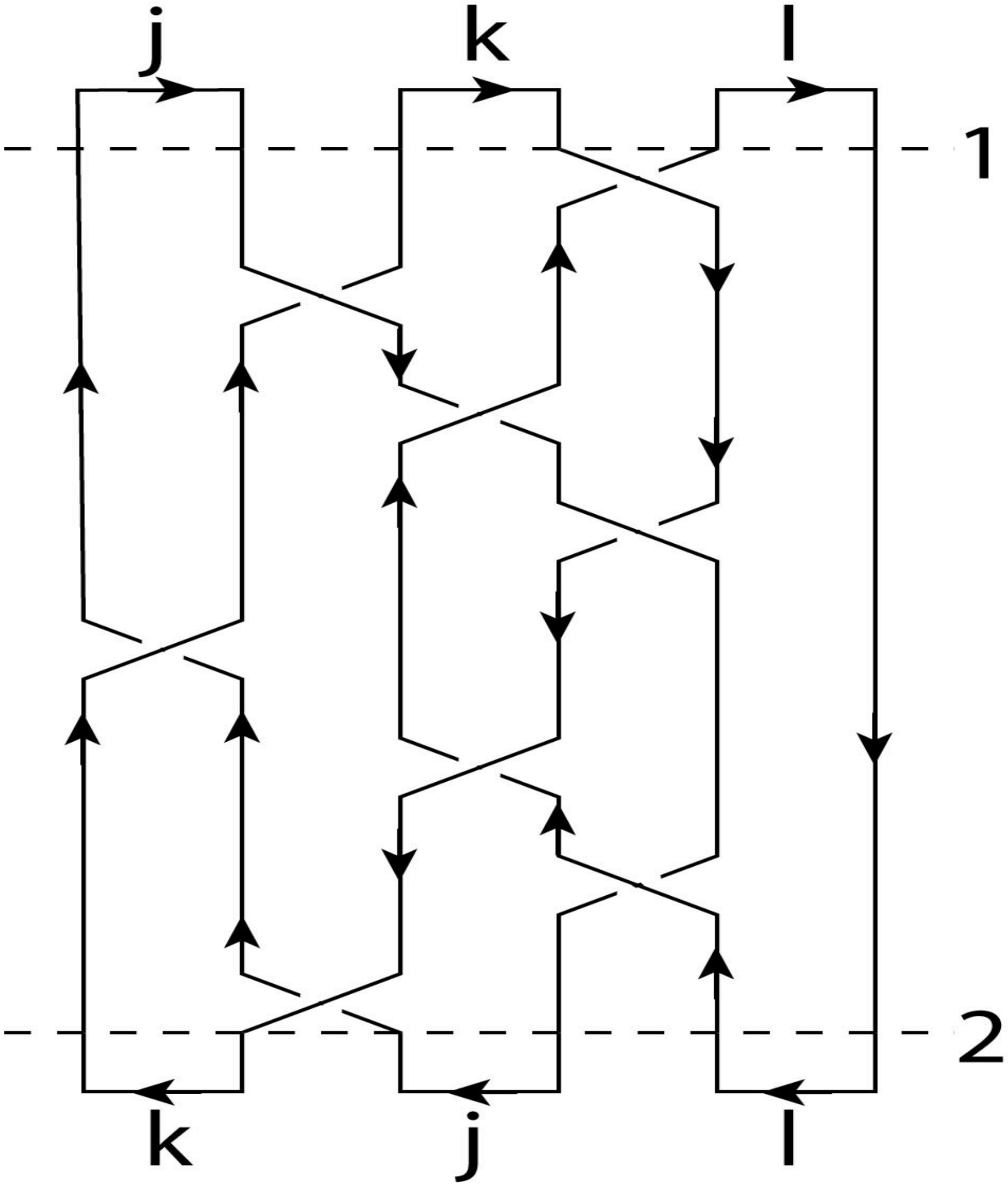

j, k, l are 'colors'
_ irreps of **SU(2)k**
3D Chern Simons
Topological Quantum
Field Theory

The orizontal levels
1 and 2 are associated with
finite dim. Hilbert spaces
H (j,k,l) and H (k,j,l)

The colored oriented braid corresponds to a unitary **Wilson operator**, whose quantum expectation value is an **observable**

Each composite Wilson operator in quantum Chern-Simons with **SU(2)k (or SU(2)q)** at any fixed root of unity **q=2_i/k**

can be split into a finite string of

1-step unitary transformations ("topological gates"):

U (_i**)**: elementary braiding operator (**B**-matrix) associated with the generator _i of the braid group

U (q-6j): q-Racah 6j-symbol or **F**usion matrix

NB Generalized, anyonic-type states can be encoded efficiently into many-qubits states and the unitaries **U (**_i**)** & **U (q-6j)** can be polynomially compiled on suitable quantum circuits

S Garnerone, A Marzuoli, M Rasetti, J Phys A: Math Theor **40** (2007)3047

Algebraic constraints for SU(2) 6j symbols **(q=1)**

Biedenharn-Elliott identity (pentagon equation)

$$\sum_x (-)^{R+x}(2x+1)\begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix}\begin{Bmatrix} c & d & x \\ e & f & q \end{Bmatrix}\begin{Bmatrix} e & f & x \\ b & a & r \end{Bmatrix} = \begin{Bmatrix} p & q & r \\ e & a & d \end{Bmatrix}\begin{Bmatrix} p & q & r \\ f & b & c \end{Bmatrix};$$

Racah identity (triangle equation: can be recasted as a hexagon equation with 3 braidings = trivial phase factors)

$$\sum_x (-)^{p+q+x}(2x+1)\begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix}\begin{Bmatrix} a & b & x \\ d & c & q \end{Bmatrix} = \begin{Bmatrix} a & c & q \\ b & d & p \end{Bmatrix},$$

$$\sum_x (2x+1)\begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix}\begin{Bmatrix} c & d & x \\ a & b & q \end{Bmatrix} = \frac{\delta_{pq}}{(2p+1)},$$

Orthogonality

Algebraic constraints for q-6j symbols **($q \neq 1$)**

Biedenharn-Elliott identity
(pentagon equation)

Same structure as before
6j _ q-6j
(2x+1) _ [2x+1] (q-dimension)

Racah identity with the substitutions: 6j _ q-6j
(2x+1) _ [2x+1] (q dimension)

can be expressed as

_ a <u>triangle</u> equation where there appear non trivial braiding factors on both sides (powers of the deformation parameter q)

_ a manifestly <u>hexagon</u> equation, by resorting to a symmetry property of the q-9j symbol ____

Quantum Yang–Baxter equation

(symmetry property of a **q-9j** symbol decomposed as

a single sum of three q-6j symbols)

$$\sum_z (-)^z\,[2z{+}1] \begin{Bmatrix} a & g & z \\ k & c & h \end{Bmatrix} \begin{Bmatrix} a & e & b \\ f & z & g \end{Bmatrix} \begin{Bmatrix} k & b & d \\ f & c & z \end{Bmatrix} \mathrm{q}^{-\mathrm{w}(\mathrm{z,h,e,d})}$$

$$= \sum_z (-)^z\,[2z{+}1] \begin{Bmatrix} k & e & z \\ f & h & g \end{Bmatrix} \begin{Bmatrix} a & z & d \\ f & c & h \end{Bmatrix} \begin{Bmatrix} a & e & b \\ k & d & z \end{Bmatrix} \mathrm{q}^{-\mathrm{w}(\mathrm{z,g,c,b})}$$

Summarizing:

TOPOLOGICAL INVARIANCE OF BRAID DIAGRAMS
translated into
ALGEBRAIC CONSTRAINTS with structures

FF = FFF (PENTAGON)

FBF = BFB (HEXAGON)

F _ U (q-6j)
B _ U (_i)

The constraints ensure fault-tolerant quantum computation since vacuum expectation values of Wilson operators, *e.g.*

< **W** = BFBFFB…BF> (finite 'strings')

do not depend on the local features of the presentation chosen for the BRAID DIAGRAM associated with **W**

_____ ('topological' protection) _____

The same ALGEBRAIC CONSTRAINTS in **$SU(2)_k$** 3D Chern-Simons provide also **solvability** of the theory for each fixed value of the coupling constant **k** (quantum expectation values are **finite** topological invariants of knots)

Issues to be addressed

_ perturbative expansions **(k >>1)** of constraints also in view of efficient computation of Vassiliev-type invariants

_ asymptotic limits of the hexagon constraint (and 3nj-symbols) & relations with 2D integrable lattice models

(a limit of the former Quantum YB equation when some entries of the symbols become large generates spectral YB for the IRF model)