

Quantum capacity of memory channels

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Motivations

- Non-Markovian effects in open quantum systems
 - Low-frequency noise noise in solid-state devices (for instance, $1/f$ noise)
 - Fluctuating birefringe in optical fibers
 - Quantum information transmission across spin chains
 - Sending atoms through a resonant cavity
- Can memory effects enhance the capacity of quantum channels?
- Connections with the physics of many-body systems (critical phenomena,...)

Outline

1) Dephasing channels with memory:

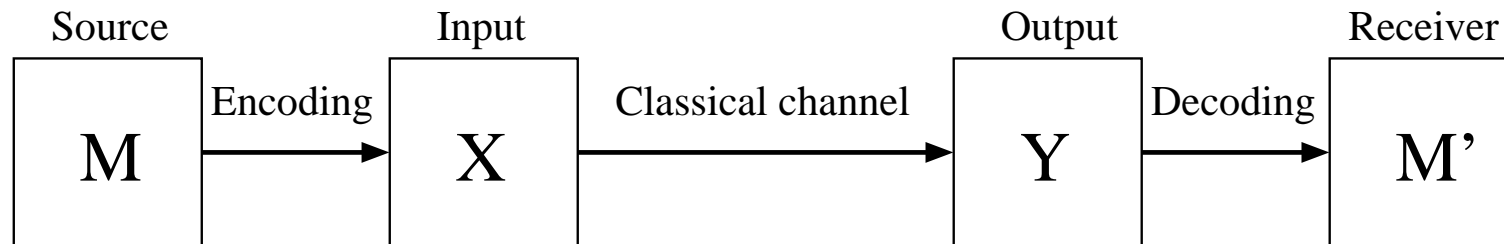
(a) Markov chain model: explicit computation of the quantum capacity

(b) Bosonic bath of oscillators: numerically obtained lower bound

2) Channels with damping:

Dissipation can help increase the transmission rate

Capacity of a classical channel



MUTUAL INFORMATION $I(X : Y) = H(X) + H(Y) - H(X, Y)$

$H(X) = - \sum_x p_x \log_2 p_x$ Shannon information of the random variable X

CAPACITY: maximum rate at which classical information can be reliably transmitted down the channel

$$C = \max_{p_x} H(X : Y)$$

Quantum channels

QUANTUM SOURCE: quantum states chosen from the ensemble $\{\rho_0, \dots, \rho_k\}$ with a priori probabilities $\{p_1, \dots, p_k\}$ are sent through the channel

QUANTUM CHANNEL described by a linear, completely positive, trace preserving (CPT) map \mathcal{E} :

$$\rho' = \mathcal{E}(\rho), \quad \rho = \sum_{x=1}^k p_x \rho_x$$

Use quantum states to reliably transmit classical information (**classical capacity**) or quantum information (**quantum capacity**)

How to measure the reliability in the transmission of quantum information?

It is not sufficient to verify that the input state ρ is transmitted with high fidelity
Ex: send a member of a Bell pair through a completely dephasing channel

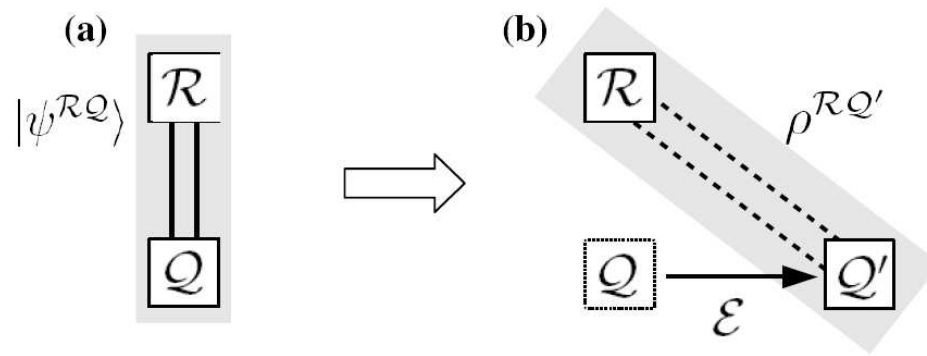
$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad \rho = \text{Tr}_2(|\psi\rangle_{12}\langle\psi|) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

$$\mathcal{E}(\rho) = \rho, \quad \text{but} \quad (\mathcal{E} \otimes \mathcal{I})(|\psi\rangle_{12}\langle\psi|) = \frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|)$$

Entanglement is lost

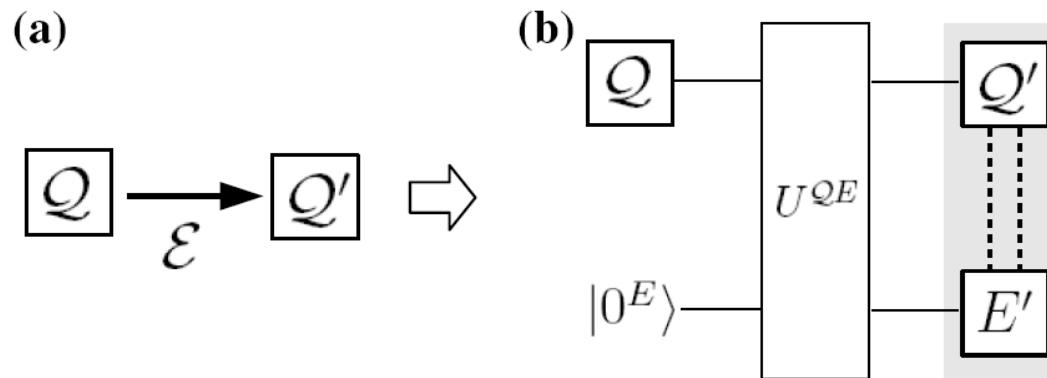
Entanglement fidelity



$$\begin{aligned}
 F_e = F_e(\rho, \mathcal{E}) &= F(|\psi^{RQ}\rangle, \rho^{RQ'}) = \langle \psi^{RQ} | \rho^{RQ'} | \psi^{RQ} \rangle \\
 &= \langle \psi^{RQ} | (\mathcal{E}^Q \otimes \mathcal{I}^R) (|\psi^{RQ}\rangle \langle \psi^{RQ}|) | \psi^{RQ} \rangle
 \end{aligned}$$

The **ENTANGLEMENT FIDELITY** F_e is independent of the purification \mathcal{R} of the quantum system Q

Entropy exchange



$$S_e = S_e(\rho, \mathcal{E}) = S(\rho^{E'}), \quad S(\rho) = -\text{Tr}(\rho \log_2 \rho) \quad \text{von Neumann entropy}$$

The **ENTROPY EXCHANGE** S_e is the entropy of the final state $\rho^{E'}$ of a 'mock' environment, initially in a pure state $|0^E\rangle$

Coherent information

Analogous to mutual information but for quantum information

$$I_c(\rho, \mathcal{E}) = S(\mathcal{E}(\rho)) - S_e(\rho, \mathcal{E})$$

The **COHERENT INFORMATION** $I_c = S(\rho^{\mathcal{Q}'}) - S(\rho^{\mathcal{R}\mathcal{Q}'})$ can never be positive for classical systems

I_c deals with the entanglement transmission through the channel

Ex: if the channel is noiseless, $I_c = 0$ if ρ is pure, I_c is maximum if ρ is maximally mixed

Quantum data-processing inequality:

$$I_c(\rho, \mathcal{E}_1) \geq I_c(\rho, \mathcal{E}_2 \circ \mathcal{E}_1)$$

We cannot increase the coherent information acting on the output

In contrast to mutual information, I_c in general is **not subadditive**

Using **entangled input states** $\rho_{12} \neq \rho_1 \otimes \rho_2$ [$\rho_1 = \text{Tr}_2(\rho_{12}), \rho_2 = \text{Tr}_1(\rho_{12})$] we can obtain

$$I_c(\rho_{12}, \mathcal{E} \otimes \mathcal{E}) > I_c(\rho_1, \mathcal{E}) + I_c(\rho_2, \mathcal{E})$$

[see Barnum, Nielsen and Schumacher, *Phys. Rev. A* **57**, 4153 (1998)]

Quantum capacity

The **QUANTUM CAPACITY** Q measures the maximum number of qubits (per channel use) that can be reliably transmitted down a noisy channel

For **memoryless channels**

$$Q = \lim_{n \rightarrow \infty} \frac{Q_n}{n}, \quad Q_n = \max_{\rho} I_c(\mathcal{E}_n, \rho)$$

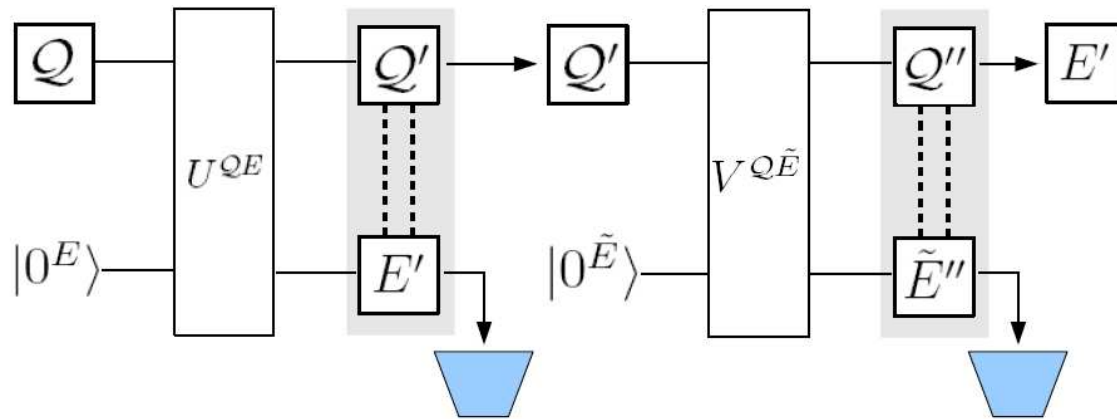
$$I_c(\mathcal{E}_n, \rho) = S[\mathcal{E}_n(\rho)] - S[\tilde{\mathcal{E}}_n(\rho)], \quad \mathcal{E}_n = \mathcal{E}^{\otimes n}$$

$$\rho' = \mathcal{E}_n(\rho) = \text{Tr}_E[U_n(\rho \otimes |0\rangle_E \langle 0|)U_n^\dagger], \quad \rho'_E = \tilde{\mathcal{E}}_n(\rho) = \text{Tr}_S[U_n(\rho \otimes |0\rangle_E \langle 0|)U_n^\dagger]$$

The **regularization** $n \rightarrow \infty$ is necessary since I_c in general fails to be subadditive

Degradable channels

Degradable channels: the final state $\rho'_{E'}$ of the environment can be reconstructed from the final state ρ' of the system



In this case $I_c = S(\rho^{Q'E'}) - S(\rho^{E'}) = S(Q'|E')$ is subadditive

$Q = Q_1$ (“single-letter” formula)

[see Devetak and Shor, *Commun. Math. Phys.* **256**, 287 (2005)]

Dephasing channels

Single-use dephasing channel:

$$\mathcal{E} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & g\rho_{01} \\ g\rho_{10} & \rho_{11} \end{pmatrix}, \quad g \text{ dephasing factor}$$

Generalized n -uses dephasing channel

$$U_n |i\rangle |0\rangle_E = |i\rangle |\phi_i\rangle_E, \quad |i\rangle = |i_1, \dots, i_n\rangle \text{ preferential basis}$$

$$\rho' = \mathcal{E}_n(\rho) = \sum_{\alpha} A_{\alpha} \rho A_{\alpha}^{\dagger}, \quad A_{\alpha} = {}_E \langle \alpha | U_n | 0 \rangle_E \text{ diagonal Kraus operators}$$

$$(A_{\alpha})_{ij} = {}_E \langle \alpha | \phi_i \rangle_E \delta_{ij}$$

In general, $\mathcal{E}_n \neq \mathcal{E}^{\otimes n}$ (MEMORY EFFECTS)

Degradability of the generalized dephasing channel

$$\rho = \sum_{i,j} c_{ij} |i\rangle\langle j| \text{ generic input state}$$

The final environmental state depends only on the populations of ρ

$$\rho'_E = \tilde{\mathcal{E}}_n(\rho) = \sum_i |c_i|^2 |\phi_i\rangle_E \langle \phi_i|$$

Since the dephasing channel \mathcal{E}_n does not affect populations, $\tilde{\mathcal{E}}_n = \tilde{\mathcal{E}}_n \circ \mathcal{E}_n$

We have proved that the coherent information $I_c(\mathcal{E}_n, \rho)$ of a generalized dephasing channel is maximized by **SEPARABLE** input states **DIAGONAL** in the preferential basis $\{|i\rangle\}$

Forgetful channels

Memory effects dies out exponentially fast with time

DOUBLE-BLOCKING strategy:

(mapping, with negligible error, into a memoryless channel)

- consider blocks of $n + l$ uses of the channel
- do the actual coding and decoding for the first n uses
- let $n \rightarrow \infty$
- quantum capacity $Q = \lim_{n \rightarrow \infty} \frac{Q_n}{n}$

[see Kretschmann and Werner, *Phys. Rev. A* **72**, 062323 (2005)]

A phenomenological noise model

$$\rho' = \mathcal{E}_n(\rho) = \sum_{i_1, \dots, i_n=0, z} A_{i_1 \dots i_n} \rho A_{i_1 \dots i_n}^\dagger, \quad A_{i_1 \dots i_n} = \sqrt{p_{i_1 \dots i_n}} \sigma_{i_1}^{(1)} \otimes \dots \otimes \sigma_{i_n}^{(n)},$$

$p_{i_1 \dots i_n}$ probability that the the ordered sequence $\sigma_{i_1}^{(1)}, \dots, \sigma_{i_n}^{(n)}$ of Pauli operators (I or σ_z) is applied to the n qubits crossing the channel

- **Dephasing probability stationary:** $p_{i_k=z} = p_z$ [$p_{i_k=0} = p_0 = 1 - p_z$] for all k
($p_{i_k} = \sum_{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n} p_{i_1 \dots i_n}$)

The maximum of coherent information in this model is obtained for the **maximally mixed input state** $\rho_I \equiv \frac{1}{2^n} I^{\otimes n}$

Quantum capacity of dephasing channels with memory,
New J. Phys. **9**, 310 (2007)

Markov-chain model

$$p_{i_1, \dots, i_n} = p_{i_1} p_{i_2|i_1} \cdots p_{i_n|i_{n-1}}, \quad p_{i_k|i_{k-1}} = (1 - \mu) p_{i_k} + \mu \delta_{i_k, i_{k-1}}$$

μ measures the **partial memory** of the channel

$\mu = 0$ memoryless channel

$\mu = 1$ perfect memory

μ might depend on the time interval τ between two consecutive channel uses,
compared with the **memory time scale** τ_c

We have proved that this model is **forgetful**

Quantum capacity of a Markov chain

$$S[\mathcal{E}_n(\rho_I)] = S(\rho_I) = n$$
$$S_e = - \sum_{i_1, \dots, i_n} p_{i_1 \dots i_n} \log_2 p_{i_1 \dots i_n} \equiv H(X_1, \dots, X_n)$$

$H(X_1, \dots, X_n)$ Shannon entropy of the collection of random variables X_1, \dots, X_n
(characterized by the joint probabilities $p_{i_1 \dots i_n}$)

For a stationary Markov chain (see Cover and Thomas, *Elements of Information Theory*)

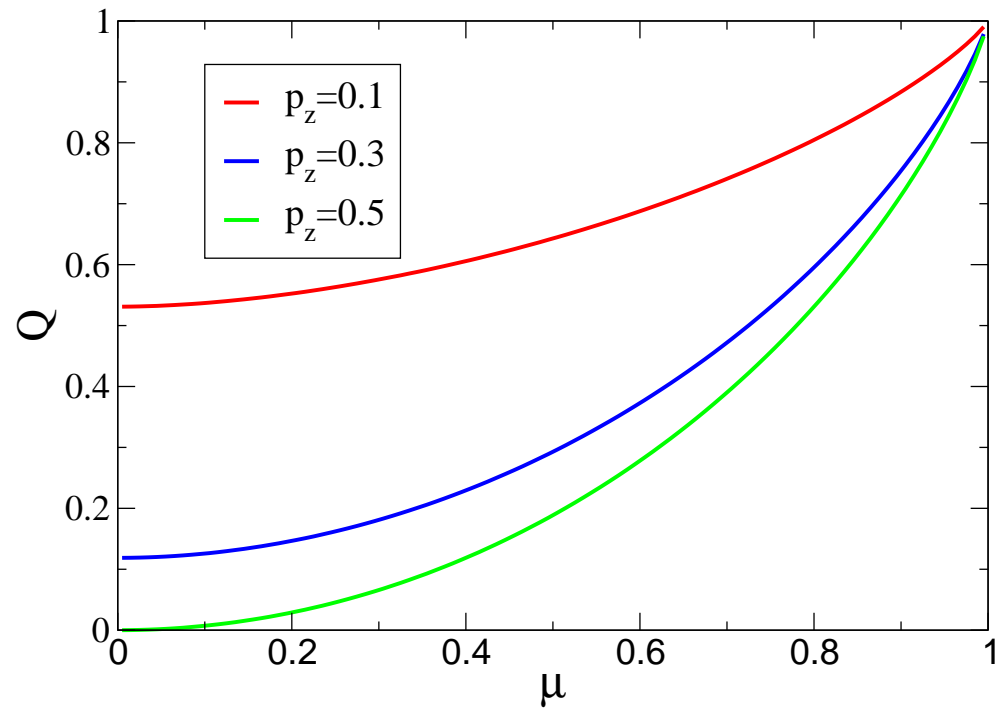
$$\lim_{n \rightarrow \infty} \frac{1}{n} H(X_1, \dots, X_n) = H(X_2|X_1) = p_0 H(q_0) + p_z H(q_z), \quad q_i \equiv p(i|i) = (1-\mu)p_i + \mu$$

Quantum capacity of dephasing channels with memory,
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$$Q = 1 - p_0 H(q_0) - p_z H(q_z)$$

$$Q = 1 - H(p_0) \text{ memoryless limit}$$

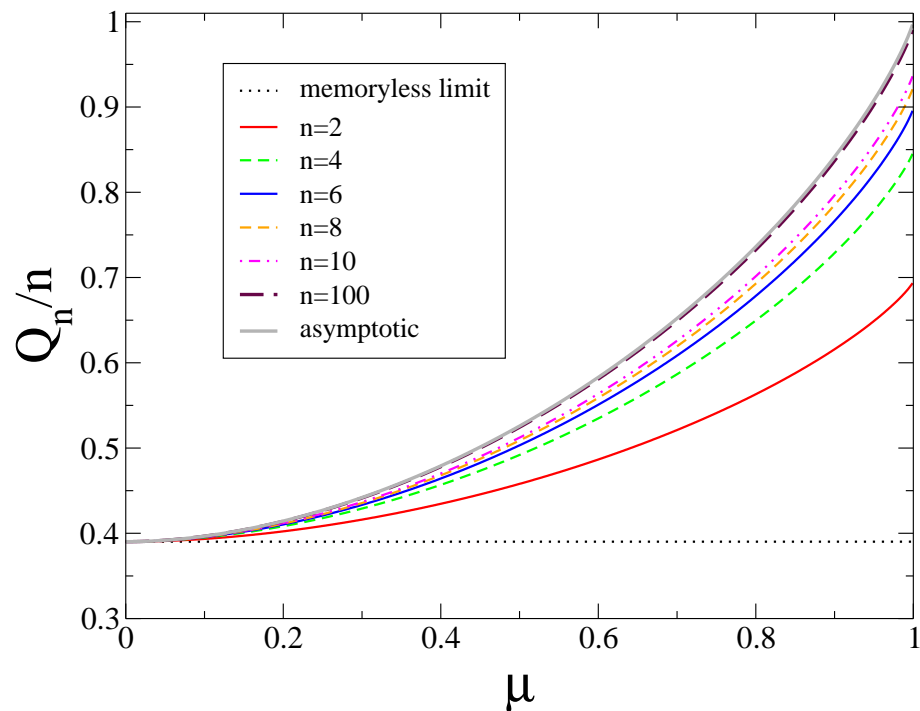
$$Q = 1 \text{ with perfect memory} \\ \text{(noiseless channel)}$$



[see also Hamada, *J. Math. Phys.* **43**, 4382 (2002); Plenio and Virmani, *Phys. Rev. Lett.* **99**, 120504 (2007)]

Convergence of Q_n/n to Q

$$Q_n = n - (n - 1)[p_0 H(q_0) + p_z H(q_z)] - H(p_0)$$



$\epsilon_n \equiv Q - \frac{Q_n}{n}$ growing function of μ

$$\epsilon_n(\mu = 0) = 0$$

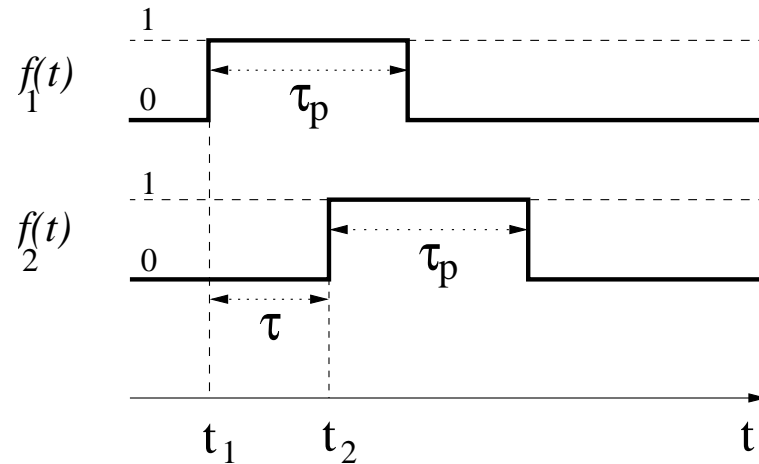
$$\epsilon_n(\mu = 1) = H(p_0)/n$$

$$\epsilon_n(\mu) \approx \frac{1}{2 \ln 2} \frac{\mu^2}{n} \text{ for } \mu \ll 1$$

Bosonic bath environment

$$H(t) = H_E - \frac{1}{2}X_E F(t) + H_C, \quad H_E = \sum_{\alpha} \omega_{\alpha} b_{\alpha}^{\dagger} b_{\alpha},$$

$$X_E = \sum_{\alpha} (b_{\alpha}^{\dagger} + b_{\alpha}), \quad F(t) = \lambda \sum_{j=1}^n \sigma_z^{(j)} f_j(t), \quad H_C = \sum_{\alpha} \frac{\lambda^2}{4\omega_{\alpha}} \sum_{j=1}^n \sigma_z^{(j)}$$



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$$\rho(t) = \text{Tr}_E[U(t)(\rho \otimes \rho_E)U^\dagger(t)], \quad U(t) = T e^{-\frac{i}{\hbar} \int_0^t ds H(s)}$$

$U(t|i) = \langle i|U(t)|i\rangle$ conditional evolution operator for the environment alone

$$(\rho')_{ij} = (\rho)_{ij} \sum_{\alpha} {}_E \langle \alpha | U(t|i) \rho_E U^\dagger(t|j) | \alpha \rangle_E$$

Multimode environment of oscillators initially at thermal equilibrium,

$\rho_E = \exp(-\beta H_E)$:

$$\sum_{\alpha} {}_E \langle \alpha | U(t|i) \rho_E U^\dagger(t|j) | \alpha \rangle_E = e^{\left\{ -\lambda^2 \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{1 - \cos(\omega\tau p)}{\omega^2} \left| \sum_{k=1}^n (i_k - j_k) e^{i\omega(k-1)\tau} \right|^2 \right\}}$$

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Assume that the bath correlation function

$$C(t) \equiv \frac{1}{2} \langle X_E(t)X_E(0) + X_E(0)X_E(t) \rangle$$

decays exponentially with time (we conjecture that the channel is forgetful)

This is the case, e.g., for a Lorentian power spectrum

$$S(\omega) = \frac{2\tau_c}{[1 + (\omega\tau_c)^2]}$$

Decoherence-protected subspace

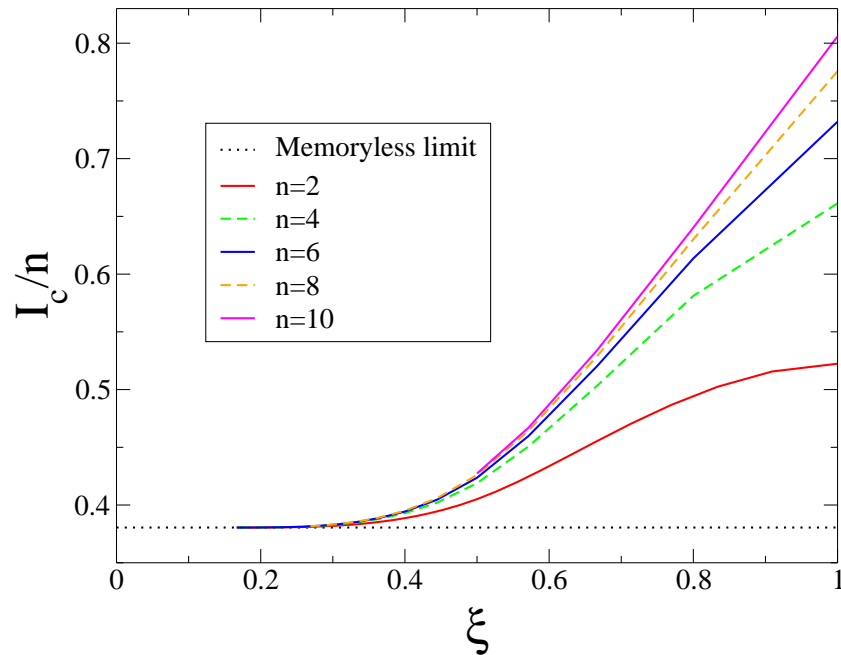
In the limit of perfect memory ($\tau_c \rightarrow \infty$) there exists for any n a decoherence-free subspace corresponding to a qubit train with an equal number of $|0\rangle$ and $|1\rangle$ states

Since the dimension d of this subspace is such that $\log_2 d \approx n - \frac{1}{2} \log_2 n$, then the channel is **asymptotically noiseless** ($Q = 1$)

If $\bar{n} \gg 1$ qubits can be sent within the memory time scale τ_c and the quantum information is encoded in the decoherence-protected subspace, then

$$1 - \log_2 \bar{n} / (2\bar{n}) \quad \text{lower bound for } Q_{\bar{n}} / \bar{n}$$

Lower bound for Q_n/n



maximally mixed input state

Lorentzian power spectrum

$$\xi \equiv \tau_c / (\tau + \tau_c)$$

Numerical data suggest that I_c/n converges for $n \rightarrow \infty$: it is possible to increase the transmission **rate** if quantum information is encoded in long blocks, separated by time intervals larger than τ_c

Memory effects in a quantum channel with damping

Qubit-oscillator interaction of the Jaynes-Cummings kind, but with a switchable coupling

$$H(t) = H_0 + V, \quad H_0 = \nu \left(a^\dagger a + \frac{1}{2} \right) + \frac{\omega}{2} \sum_{k=1}^n \sigma_z^{(k)},$$

$$V = \lambda \sum_{k=1}^n f_k(t) \left(a^\dagger \sigma_-^{(k)} + a \sigma_+^{(k)} \right)$$

$f_k(t) = 1$ when the k -th qubit is inside the channel

$f_k(t) = 0$ otherwise

Resonance condition: $\omega = \nu$

Memoryless limit

Generic single-qubit input state $\rho_S = (1 - p)|g\rangle\langle g| + r|g\rangle\langle e| + r^*|e\rangle\langle g| + p|e\rangle\langle e|$

Oscillator at each channel use prepared in its ground state $\rho_E = |0\rangle\langle 0|$

$$\text{Output state } \rho'_S = \text{Tr}_E[\rho'_{SE}] = \begin{bmatrix} 1 - p \cos^2(\Omega\tau_p) & r \cos(\Omega\tau_p) \\ r^* \cos(\Omega\tau_p) & p \cos^2(\Omega\tau_p) \end{bmatrix}, \quad \tau_p \text{ crossing time}$$

We have an amplitude damping channel

$$\rho'_S = \sum_k E_k \rho_S E_k^\dagger, \quad E_0 = |g\rangle\langle g| + \sqrt{1 - \eta}|e\rangle\langle e|, \quad E_1 = \sqrt{\eta}|g\rangle\langle e|, \quad \eta \equiv \sin^2(\Omega\tau_p)$$

This channel is degradable and its quantum capacity is known [see Giovannetti and Fazio, PRA **71**, 032314 (2005)]

Dissipative effects

We describe dissipation by means of the master equation

$$\dot{\rho}_E = \Gamma \left(a\rho_E a^\dagger - \frac{1}{2}a^\dagger a\rho_E - \frac{1}{2}\rho_E a^\dagger a \right)$$

Assume dephasing time τ_ϕ much shorter than the time τ between two channel uses: coherences die out before a new qubit enters the channel

Relevant time scales:

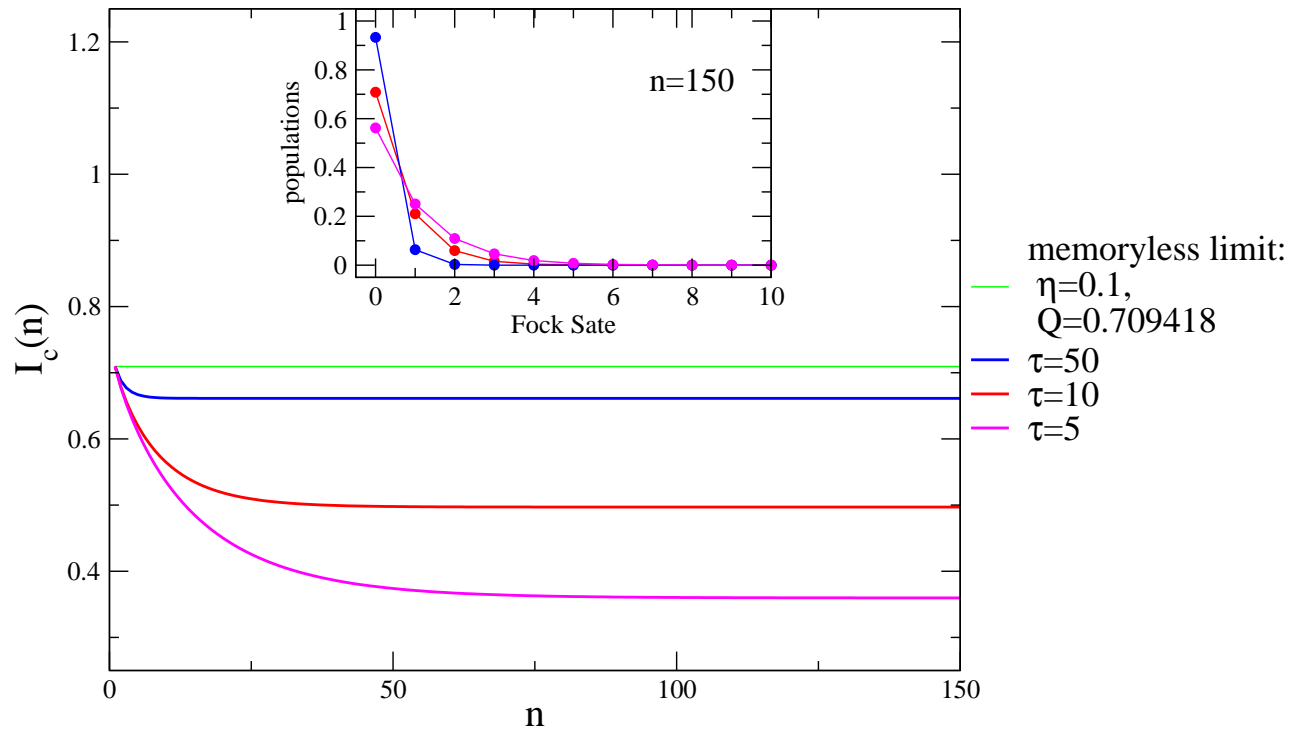
$$\tau_p \ll \tau_\phi \ll \tau \ll \tau_d$$

$\tau \ll \tau_d$: memory effects are relevant

Non-equilibrium steady state and coherent information

Amplitude Damping Channel: $\eta=0.1$ ($\tau_p=0.32175055$) $\rightarrow p_{\text{opt}}=0.463058$

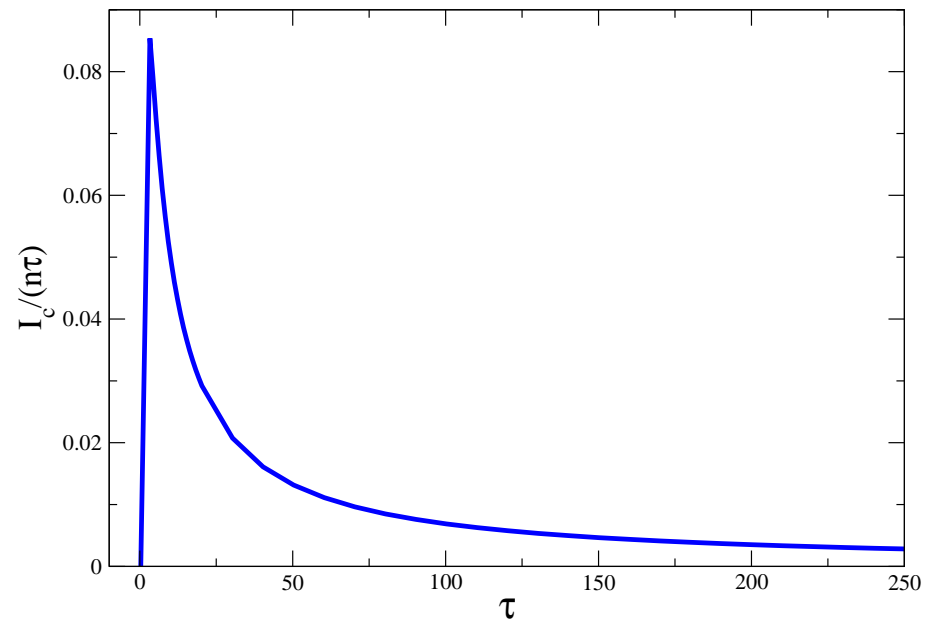
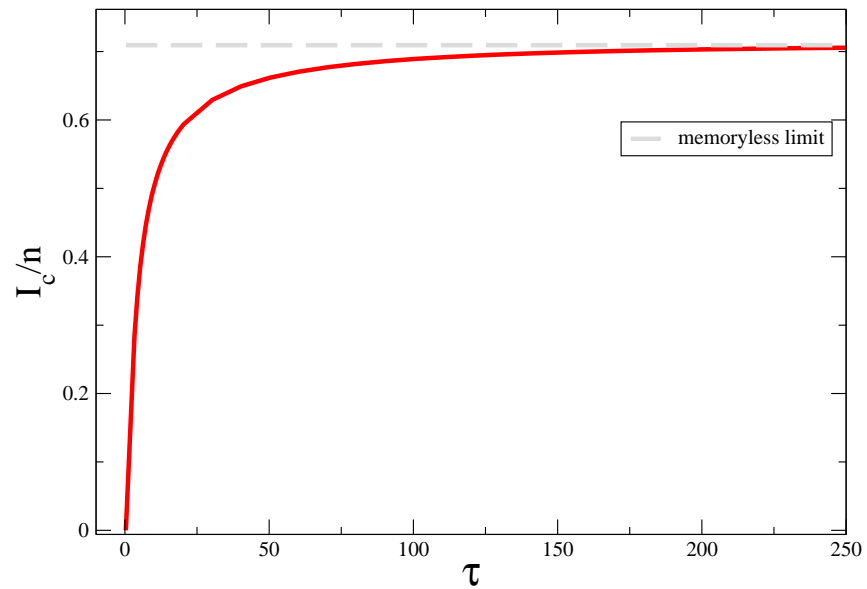
$\omega=1$; initial state environment: $|0\rangle\langle 0|$, damping rate: $\Gamma=0.01$; Qubit state: $r=0$



Dissipation-induced enhancement of transmission rate

Amplitude Damping Channel: $\eta=0.1$ ($\tau_p=0.32175055$) $\rightarrow p_{\text{opt}}=0.463058$

$\omega=1$; initial state environment: $|0\rangle\langle 0|$, damping rate: $\Gamma=0.01$; qubit state: $r=0$; $n=200$



Conclusions

Memory effects can be exploited to enhance the **transmission rate** of quantum information

This result might be relevant for quantum state transmission, entanglement distribution and quantum error correction

- Find realistic coding strategies for few-qubit trains (see D'Arrigo, De Leo, Benenti, Falci, arXiv:0802.4172)
- The results of this study could be adapted to environments with algebraically decaying memory effects, e.g. **low-frequency noise in the solid state?**
- Investigate the effects of integrable/chaotic environments and of quantum phase transitions on quantum channel capacity