On the quantum theory of measurements continuous in time: information gain[a]

Camerino, October 2008 — Joint work with G. Lupieri

- Quantum continuous measurements = Quantum trajectories: a quantum system is taken under observation with continuity in time (the output is not a single random variable, but a stochastic process)
- Aim: to characterize the behaviour of the measurement, to quantify its effectiveness in extracting information from the quantum system by means of mutual entropies

  Mutual entropy: relative entropy of a (classical, quantum, or mixed) state on a product algebra (bipartite system) with respect to the product of its marginals
- Key concept: any quantum measurement is a "quantum channel", a CP map from the quantum states (density operators) to classical/quantum states (probabilities for the output & post-measurement density operator)

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The linear SDE of continual measurement theory.

For simplicity: bounded operators, finite sums...

$\mathcal{H}$: Hilbert space of the system $\mathcal{T}(\mathcal{H})$: trace class

$\mathcal{S}(\mathcal{H}) = \{\text{statistical operators}\} \subset \mathcal{T}(\mathcal{H})$ Initial condition: $\sigma_0 = \varrho \in \mathcal{S}(\mathcal{H})$

$$\mathrm{d}\sigma_t = \mathcal{L}(t)[\sigma_t]\mathrm{d}t + \sum_j \left(R_j(t)\sigma_t + \sigma_t R_j(t)^*\right) \mathrm{d}W_j(t) + \sum_k \left(\frac{\mathcal{J}_k(t)[\sigma_t]}{\lambda_k} - \sigma_t \right) \left(\mathrm{d}N_k(t) - \lambda_k \mathrm{d}t\right)$$

$W_j, j = 1, \ldots$, $N_k, k = 1, \ldots$: independent Wiener and Poisson processes in a probability space $(\Omega, \mathcal{F}^0_\infty, \mathbb{Q})$. $\mathbb{E}_\mathbb{Q}[N_k(t)] = \lambda_k t$: intensity $\lambda_k > 0$

The space of events from s to t: the σ-algebra generated by the increments $\mathcal{F}^s_t = \sigma\{W_j(u) - W_j(s),\ N_k(v) - N_k(s),\ u, v \in [s, t],\ j, k = 1, \ldots\}$

$\mathcal{L}(t)$: Liouville operator $\mathcal{J}_k(t)$: jump operator

$$\mathcal{J}_k(t)[\rho] = \sum_r V_k^r(t)\rho V_k^r(t)^* \qquad J_k(t) = \mathcal{J}_k(t)^*[\mathbb{1}] = \sum_r V_k^r(t)^* V_k^r(t)$$

$$\mathcal{L}(t)[\rho] = -\mathrm{i}[H(t),\rho] + \sum_l \left(L_l(t)\rho L_l(t)^* - \frac{1}{2}\left\{ L_l(t)^* L_l(t), \rho \right\} \right)$$
$$+ \sum_j \left(R_j(t)\rho R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \rho \right\} \right) + \sum_k \left(\mathcal{J}_k(t)[\rho] - \frac{1}{2}\left\{ J_k(t), \rho \right\} \right)$$

$H(t)$, $L_l(t)$, $R_j(t)$, $V_k^r(t)$ are bounded operators with $H(t) = H(t)^*$ and with a *regular* dependence on time (norm-continuity from the left with limits from the right)

Physical probabilities and a posteriori states

$\sigma_0 = \varrho \in \mathcal{S}(\mathcal{H}) \quad \Rightarrow \quad \sigma_t(\omega) \geq 0\,,\ \int_\Omega \mathrm{Tr}\{\sigma_t(\omega)\}\mathbb{Q}(\mathrm{d}\omega) = 1$

$p_t := \mathrm{Tr}\{\sigma_t\}$ is a martingale; $\mathbb{P}_t(\mathrm{d}\omega) = p_t(\omega)\mathbb{Q}(\mathrm{d}\omega)\big|_{\mathcal{F}_t^0}$ is a family of consistent probabilities (the physical probabilities), i.e. $F \in \mathcal{F}_t^0\,,\ t \leq T \Rightarrow \mathbb{P}_T[F] = \mathbb{P}_t[F]$.

a posteriori states: $\rho_t := \frac{1}{p_t}\sigma_t$ a priori states: $\eta_t := \mathbb{E}_{\mathbb{Q}}[\sigma_t] = \mathbb{E}_{\mathbb{P}_t}[\rho_t]$

ρ_t satisfies a non-linear SDE (the stochastic master equation) $\rho_0 = \eta_0 = \sigma_0 = \varrho$

In quantum information $\sigma_t \equiv \{P_t, \rho_t\}$ is called an ensemble (of quantum states) with average state η_t

$S_{\mathrm{q}}(\rho_t \| \eta_t) := \mathrm{Tr}\{\rho_t(\ln \rho_t - \ln \eta_t)\}$: a quantum relative entropy

σ_t is a normal state on the von Neumann algebra $L^\infty\big(\Omega, \mathcal{F}_t^0, \mathbb{Q}; \mathcal{B}(\mathcal{H})\big) \simeq L^\infty\big(\Omega, \mathcal{F}_t^0, \mathbb{Q}\big) \otimes \mathcal{B}(\mathcal{H})$ (a bipartite classical/quantum system)

The relative entropy of the ensemble σ_t with respect to the product of its marginals p_t and η_t:

$$S(\sigma_t \| p_t \eta_t) := \int_\Omega \mathbb{Q}(\mathrm{d}\omega)\ \mathrm{Tr}\{\sigma_t(\omega)(\ln \sigma_t(\omega) - \ln p_t(\omega)\eta_t)\} \equiv \mathbb{E}_{\mathbb{P}_t}[S_{\mathrm{q}}(\rho_t \| \eta_t)]$$

it is a mutual entropy of mixed classical/quantum type — Holevo's χ-quantity of the ensemble σ_t — It is a measure of the effectiveness of the continual measurement in encoding information in the a posteriori states

Instruments and channels. General result: any instrument is equivalent to a channel from quantum states to classical/quantum states.

The propagator Λ_t^s of the linear SDE: $\qquad \varrho \mapsto \sigma_t = \Lambda_t^0[\varrho]$

$\{u_j\}$ c.o.n.s. in $\mathcal{H}$, $\quad \Lambda_t^s[|u_i\rangle\langle u_j|] = |u_i\rangle\langle u_j| + \int_s^t \mathcal{L}(r)\circ\Lambda_r^s[|u_i\rangle\langle u_j|]\mathrm{d}r +$

$$+\sum_j \int_s^t \Big(R_j(r)\Lambda_r^s[|u_i\rangle\langle u_j|] + \Lambda_r^s[|u_i\rangle\langle u_j|]R_j(r)^*\Big)\mathrm{d}W_j(r)$$

$$+\sum_k \int_s^t \left(\frac{\mathcal{J}_k(r)}{\lambda_k} - \mathbb{1}\right)\circ\Lambda_r^s[|u_i\rangle\langle u_j|]\left(\mathrm{d}N_k(r) - \lambda_k\mathrm{d}r\right)$$

$\Lambda_t^s : \mathcal{T}(\mathcal{H}) \to L^1\big(\Omega, \mathcal{F}_t^s, \mathbb{Q}; \mathcal{T}(\mathcal{H})\big)$ is CP and normalized: it is a channel

For $0 \leq r \leq s \leq t$, $\qquad \Lambda_t^s \circ \Lambda_s^r = \Lambda_t^r, \qquad \Lambda_t^s[\sigma_s] = \sigma_t$

For $F \in \mathcal{F}_t^s$ define $\mathcal{I}_t^s(F)$: $\quad \forall \rho \in \mathcal{S}(\mathcal{H}), \quad \mathcal{I}_t^s(F)[\rho] = \mathbb{E}_{\mathbb{Q}}\left[1_F\, \Lambda_t^s[\rho]\right]$

$\mathcal{I}_t^s(\bullet)$ is an instrument, a normalized CP-map valued measure

$\mathcal{I}_t^s(\bullet)^*[\mathbb{1}]$ is a POV measure (a general observable in quantum mechanics)

Physical probabilities, output of the measurement; Girsanov theorem

Structure of the probability density $p_t = \mathrm{Tr}\{\sigma_t\}$:

$$p_t = \exp\Bigg\{\sum_j \bigg[\int_0^t m_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2\,\mathrm{d}s\bigg] + \sum_k \bigg[\int_0^t \ln\frac{\mu_k(s)}{\lambda_k}\,\mathrm{d}N_k(s) + \int_0^t (\lambda_k - \mu_k(s))\,\mathrm{d}s\bigg]\Bigg\}$$

A posteriori means : $\quad m_j(t) = \mathrm{Tr}\left\{(R_j(t) + R_j(t)^*)\,\rho_t\right\} \qquad \mu_k(t) = \mathrm{Tr}\left\{J_k(t)\rho_t\right\}$

Output of the measurement: the processes $W_j(t)$, $N_k(t)$ under the physical probability

Under the physical probability $\mathbb{P}_T(\mathrm{d}\omega) = p_T(\omega)\mathbb{Q}(\mathrm{d}\omega)\big|_{\mathcal{F}_T^0}$ the processes $\widehat{W}_j(t) = W_j(t) - \int_0^t m_j(s)\,\mathrm{d}s$ $(0 \leq t \leq T)$ are independent, standard Wiener processes and $N_k(t)$ is a counting process of stochastic intensity $\mu_k(t)\mathrm{d}t$.

A priori means: $\begin{cases} n_j(t) := \mathbb{E}_{\mathbb{P}_t}[m_j(t)] = \mathrm{Tr}\left\{(R_j(t) + R_j(t)^*)\,\eta_t\right\} \\ \nu_k(t) := \mathbb{E}_{\mathbb{P}_t}[\mu_k(t)] = \mathrm{Tr}\left\{J_k(t)\eta_t\right\} \end{cases}$

Aim: to introduce a reference measure with density q_t such that $S_c(p_t\|q_t) = \mathbb{E}_{\mathbb{P}_t}[\ln(p_t/q_t)]$ be a measure of the effectiveness of the continual measurement in extracting information on the underlying quantum system. Candidate:

$$q_t = \exp\Bigg\{\sum_j\bigg[\int_0^t n_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t n_j(s)^2\,\mathrm{d}s\bigg] + \sum_k\bigg[\int_0^t \ln\frac{\nu_k(s)}{\lambda_k}\,\mathrm{d}N_k(s) + \int_0^t (\lambda_k - \nu_k(s))\,\mathrm{d}s\bigg]\Bigg\}$$

Under $q_T(\omega)\mathbb{Q}(\mathrm{d}\omega)$, the processes W_j, N_k have independent increments as under $\mathbb{Q}$ (so, they can be interpreted as noises), but the means have been changed and made equal to the means they have under $\mathbb{P}_T$. Precisely, the processes $W_j(t) - \int_0^t n_j(s)\,\mathrm{d}s$ are independent, standard Wiener processes and $N_k(t)$ is a Poisson process of time dependent intensity $\nu_k(t)$.

In some sense $q_t(\omega)\mathbb{Q}(\mathrm{d}\omega)$ is a continuous product of marginals of $\mathbb{P}_t(\mathrm{d}\omega) = p_t(\omega)\mathbb{Q}(\mathrm{d}\omega)$ and the classical relative entropy $S_c(p_t\|q_t)$ can be considered as a mutual entropy.

Explicit computations of $S_{\mathrm{c}}(p_t\|q_t) = \mathbb{E}_{\mathbb{P}_t}[\ln(p_t/q_t)]$:

$$S_{\mathrm{c}}(p_t\|q_t) = \frac{1}{2}\sum_j \int_0^t \mathrm{Var}_{\mathbb{P}_t}[m_j(s)]\mathrm{d}s + \sum_k \int_0^t \mathbb{E}_{\mathbb{P}_t}\left[\mu_k(s)\ln\frac{\mu_k(s)}{\nu_k(s)}\right]\mathrm{d}s$$

A bound on the rate of information which can be extracted

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(S_{\mathrm{c}}(p_t\|q_t) + \mathbb{E}_{\mathbb{P}_t}[S_{\mathrm{q}}(\rho_t\|\eta_t)]\Big) \leq \frac{\mathrm{d}}{\mathrm{d}r}\,\mathbb{E}_{\mathbb{P}_r}[S_{\mathrm{q}}(\rho_r^t\|\eta_r)]\Big|_{r=t^+}$$

$S_{\mathrm{c}}(p_t\|q_t)$: classical information on the measured quantum system
$\mathbb{E}_{\mathbb{P}_t}\left[S_q\left(\rho_t\middle\|\eta_t\right)\right] = S(\sigma_t\|p_t\eta_t)$: information contained in the a posteriori states
$\rho_r^t := \left(\mathrm{Tr}\left\{\sigma_r^t\right\}\right)^{-1}\sigma_r^t$ is the a posteriori state of a continual measurement starting at time t from the state η_t. $\sigma_r^t := \Lambda_r^t[\eta_t]$. The r.h.s. of the bound is a measure of the ability of the continual measurement of starting a demixture of the a priori states η_t.

The classical quantity $\frac{\mathrm{d}}{\mathrm{d}t}\,S_{\mathrm{c}}(p_t\|q_t)$ is bounded by the quantum quantity $\frac{\mathrm{d}}{\mathrm{d}r}\,\mathbb{E}_{\mathbb{P}_r}[S_{\mathrm{q}}(\rho_r^t\|\eta_r)]\Big|_{r=t^+} - \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{\mathbb{P}_t}[S_{\mathrm{q}}(\rho_t\|\eta_t)]$

Gain of information on the initial state: the input/output classical information

Possible initial states: $\rho_{\mathrm{i}}(\alpha) \in \mathcal{S}(\mathcal{H})$, $\alpha \in A$, with probability distribution $\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)$; equivalently, $\big(\rho_{\mathrm{i}}(\alpha),\, \mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)\big)$ is the initial ensemble
average initial state: $\eta_{\mathrm{i}} = \int_A \mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)\rho_{\mathrm{i}}(\alpha)$

$$\mathbb{P}_t(\mathrm{d}\omega|\alpha) = p_t(\omega|\alpha)\mathbb{Q}(\mathrm{d}\omega) \qquad p_t(\omega|\alpha) = \mathrm{Tr}\left\{\Lambda_t^0(\omega)[\rho_{\mathrm{i}}(\alpha)]\right\}$$

$$\mathbb{P}_t(\mathrm{d}\omega) = p_t(\omega)\mathbb{Q}(\mathrm{d}\omega) \qquad p_t(\omega) = \int_A p_t(\omega|\alpha)\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha) = \mathrm{Tr}\left\{\Lambda_t^0(\omega)[\eta_{\mathrm{i}}]\right\}$$

$$\mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega) = \mathbb{P}_t(\mathrm{d}\omega|\alpha)\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha) = p_t(\omega|\alpha)\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)\mathbb{Q}(\mathrm{d}\omega)$$

$$\mathbb{P}_t(\mathrm{d}\alpha|\omega) = \frac{\mathbb{P}_t(\mathrm{d}\omega|\alpha)\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)}{\mathbb{P}_t(\mathrm{d}\omega)} = \frac{p_t(\omega|\alpha)}{p_t(\omega)}\,\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)$$

$$I(t) = \int_{A\times\Omega} \mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega)\, \ln \frac{\mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega)}{\mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha)\mathbb{P}_t(\mathrm{d}\omega)} = \int_A \mathbb{P}_{\mathrm{i}}(\mathrm{d}\alpha) \int_\Omega \mathbb{P}_t(\mathrm{d}\omega|\alpha)\, \ln \frac{p_t(\omega|\alpha)}{p_t(\omega)}$$

$$\rho_t^\alpha(\omega) = \frac{\Lambda_t^0(\omega)[\rho_{\rm i}(\alpha)]}{\mathrm{Tr}\left\{\Lambda_t^0(\omega)[\rho_{\rm i}(\alpha)]\right\}} \qquad \text{a posteriori state starting from } \rho_{\rm i}(\alpha)$$

$$\rho_t(\omega) = \frac{\Lambda_t^0(\omega)[\eta_{\rm i}]}{\mathrm{Tr}\left\{\Lambda_t^0(\omega)[\eta_{\rm i}]\right\}} \qquad \text{a posteriori state starting from } \eta_{\rm i}$$

$$m_j^\alpha(t,\omega) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho_t^\alpha(\omega)\right\} \qquad m_j(t,\omega) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho_t(\omega)\right\}$$

$$\mu_k^\alpha(t,\omega) = \mathrm{Tr}\left\{J_k(t)\rho_t^\alpha(\omega)\right\} \qquad \mu_k(t,\omega) = \mathrm{Tr}\left\{J_k(t)\rho_t(\omega)\right\}$$

$$I(t) = \int_{A\times\Omega} P_t(\mathrm{d}\alpha \times \mathrm{d}\omega) \int_0^t \mathrm{d}s \Bigg\{ \frac{1}{2}\sum_j \left(m_j^\alpha(s,\omega) - m_j(s,\omega)\right)^2 + \sum_k \mu_k^\alpha(s,\omega) \ln\frac{\mu_k^\alpha(s,\omega)}{\mu_k(s,\omega)} \Bigg\}$$

The bound (Holevo, Yuen-Ozawa, Schumacher-Westmoreland-Wootters, Jacobs, Barchielli-Lupieri):

$$0 \leq I(t) \leq \chi(0) - \chi(t)$$

$$\chi(t) = \int_\Omega \mathbb{P}_t(\mathrm{d}\omega) \int_A \mathbb{P}_t(\mathrm{d}\alpha|\omega)\, S_\mathrm{q}\big(\rho_t^\alpha(\omega)\|\rho_t(\omega)\big)$$

$\int_A \mathbb{P}_t(\mathrm{d}\alpha|\omega)\, S_\mathrm{q}\big(\rho_t^\alpha(\omega)\|\rho_t(\omega)\big)$ is a random chi-quantity; then, $\chi(t)$ is a mean chi-quantity

$$\Rightarrow \chi(0) = \int_A \mathbb{P}_\mathrm{i}(\mathrm{d}\alpha)\, S_\mathrm{q}\big(\rho_\mathrm{i}(\alpha)\|\eta_\mathrm{i}\big) \qquad \text{(Holevo's chi-quantity of the initial ensemble)}$$

Proof of all the bounds: Instruments = channels & Uhlmann monotonicity theorem (channels decrease the relative entropies)

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